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Three applications of scaling to inhomogeneous, anisotropic turbulence

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The energy spectrum in three examples of inhomogeneous, anisotropic turbulence, namely, purely mechanical wall turbulence, the Bolgiano-Obukhov cascade, and helical turbulence, is analyzed. As one could expect, simple dimensional reasoning leads to incorrect results and must be supplemented by information on the dynamics. In the case of wall turbulence, a hypothesis of Kolmogorov cascade, starting locally from the gradients in the mean flow, produces an energy spectrum that obeys the standard $k^{-5/3}$ law only for $kx_3>1$, with x_3 the distance from the wall, and an inverse power law for $kx_3<1$. An analysis of the energy budget for turbulence in stratified flows shows the unrealizability of an asymptotic Bolgiano scaling. Simulation with a Gledzer-Ohkitani-Yamada shell model leads instead to a $k^{-\alpha}$ spectrum for both temperature and velocity, with $\alpha\simeq 2$, and a cross correlation between the two vanishing at large scales. In the case of non-reflection-invariant turbulence, closure analysis suggests that a purely helical cascade, associated with a $k^{-7/3}$ energy spectrum cannot take place, unless external forcing terms are present at all scales in the Navier-Stokes equation. [S1063-651X(98)08002-7]

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I. INTRODUCTION

Turbulence in nature is always inhomogeneous; the reason is the origin of the fluctuations, either in the instability of flow patterns, or in the presence in some finite volume, of temperature gradients, chemical reactions, or external stirring. If the Reynolds number is large, however, there are turbulent fluctuations at scales much smaller than that of the forcing, and to them, the idealization known as homogeneous isotropic turbulence can be applied.

In practical applications, what one is interested in is the effect of turbulence on the mean flow and on transport, which is parametrized in terms of eddy viscosities and diffusivities (see, e.g., [1]). In some cases, in order to calculate these quantities, some information on the turbulent energy spectrum is necessary, and Kolmogorov scaling [2] is usually assumed. For instance, in the derivation of Lagrangian diffusion models [3,4], the Kolmogorov scaling hypothesis is present explicitly through the assumption of Markovian velocity increments, at time scales below that of the energy containing eddies.

What becomes necessary then is some matching condition at the transition from the inertial range (small scale) to the energy containing range (forcing scale); this is essentially the problem of connecting a region of $k^{-5/3}$ scaling to the peak in the energy spectrum. Phenomenological theories have dealt with this problem [5]; more recently, the question of how fast the effect of inhomogeneity decays at small scales has been put under examination both theoretically and using reduced models [6,7].

There are situations, however, in which the problem of how far the inertial range preserves memory of the inhomogeneity of the forcing, becomes particularly serious.

The most obvious way this can happen is when forcing takes place at all scales. Notice that this does not necessarily require the presence, for example, of obstacles of corresponding sizes in the flow. Already in the case of wall turbulence [5] one has mean flow gradients at lengths ranging from the viscous range to the height of the boundary layer, which leads to a situation of coexisting ''energy range'' and ''inertial range'' eddies, distributed at all scales.

An extended forcing range develops clearly, also in the presence of stratification, due to the effect of buoyancy. In this case an additional scale, the Obukhov length [8], marking the transition from mainly mechanical to convection dominated turbulence, becomes important, and the way in which mechanical and convective contributions to the dynamics balance one another makes a description based on scaling rather nontrivial.

A third way in which the small scale dynamics of turbulence could be modified by processes in the energy range is when helicity is fed, together with energy, into the system. The importance of this process has been discussed recently by Yakhot [7], in the case of shear turbulence. Since the Navier-Stokes nonlinearity conserves both helicity and energy, one wonders whether there could exist situations characterized by a helicity cascade, analogous to the entrophy cascade of two-dimensional turbulence [9,10].

All this neglects the presence of coherent structures and intermittency, which make an approach based on scaling and a hypothesis of homogeneous and isotropic inertial range questionable, even in the idealized case of spatially homogeneous large scales [11–13]. The importance of hairpin vortices in wall turbulence [14,15] and, even more, of plumes and effects at the boundary, in convective turbulence [16–18] is well known. Helicity, on the other hand, has long been suspected to play an important role in triggering intermittency in homogeneous turbulence [19,20].

It should be mentioned, however, that with the exception of convective turbulence, intermittency and coherent structures seem to produce only minimal effects on energy spectra and transport coefficients. For this reason, they are not very interesting when it comes to deriving turbulent models for engineering applications.

The purpose of this paper is to study the behavior of the energy spectrum in the three examples of inhomogeneous turbulence listed. Dimensional analysis must necessarily be supplemented by information on the dynamics, to produce acceptable results. In wall turbulence, this will take the form of hypotheses on the distribution of vortices and on the way they are generated. In the case of convective turbulence, an analysis of the energy budget in the Navier-Stokes and the temperature equations becomes necessary to verify the realizability of different scaling hypotheses. In helical turbulence, the same task is realized by means of closure analysis. In the next section, wall turbulence is analyzed, assuming that at any height a Kolmogorov cascade is generated with the integral scale equal to the height in examination. In Sec. III, an analysis of the various possibilities for scaling in ''homogeneous'' convective turbulence is carried out, using also results from simulations of a Gledzer-Ohkitani-Yamada (GOY) shell model of the type introduced by Jensen *et al.* [21], plus buoyancy couplings. The scaling predicted by Bolgiano [22,23,8], in particular, is taken under examination. Section IV is devoted to an analysis of helical turbulence using an eddy damped quasinormal Markovian (EDQNM) closure [24,10]. Section V contains the conclusions.

II. WALL TURBULENCE

To fix the ideas imagine a turbulent flow parallel to a horizontal plane, characterized by a height δ and a stress at the surface (for unitary fluid density) v_*^2. If the Reynolds number $\mathrm{Re}=v_*\delta/\nu$ is very large, with ν the fluid viscosity, the mean velocity $\mathbf{V}$ will obey to a very good degree of approximation the logarithmic profile law [5]

$$V_1(x_3)=\frac{v_*}{\kappa}\ln\left(\frac{v_* x_3}{\beta\nu}\right),\quad r_0\ll x_3\ll\delta, \tag{1}$$

where κ and β are dimensionless constants depending on the roughness of the wall, $r_0=\nu/v_*$ is the flow inner length, and x_3 is the distance from the wall. The law of the wall, Eq. (1), does not provide information about turbulent fluctuations beyond what could be obtained from a mixing length approximation, indicating by $\mathbf{v}$ the turbulent velocities

$$\langle v_1 v_3\rangle=-\nu_T\frac{\partial V_1}{\partial x_3},\quad \nu_T=l v_T, \tag{2}$$

where the eddy size l, the characteristic turbulent velocity v_T, and the eddy viscosity ν_T all depend on the height x_3. In a scale invariant situation, using Eqs. (1) and (2),

$$l(x_3)\sim x_3 \quad\text{and}\quad v_T\sim v_*, \tag{3}$$

which means simply that gradients of strength $x_3^{-1}v_*$ at scale x_3 lead to vortices of size x_3 and characteristic velocity v_*. If one imagines that these vortices generate Kolmogorov cascades, spatially localized in height, some idea of the behavior of the energy spectrum can be obtained.

This idea is not totally unreasonable, since, during the time it takes the energy to be transferred to the viscous range, that is of the order of an integral time, eddies will move at most by a distance of the order of an integral length $\sim x_3$.

Consider then the following picture (see Fig. 1): at a given height x_3, eddies of size $l_0\sim x_3$ are generated by instability of the mean flow. These ''mother eddies'' split into ''daughter eddies'' of size $l_{0\sigma}<l_0$, producing a Kolmogorov cascade; the index σ indicates the point in the cascade and is the logarithm of the ratio of the size of the daughter eddy to that of the original mother eddy. To these, however, there

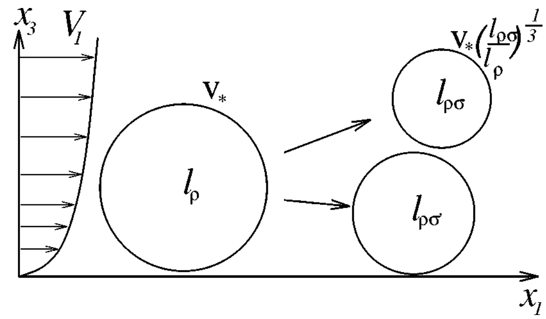

FIG. 1. Generation of Kolmogorov-like cascade; for $\rho>0$, l_ρ is larger than the reference height x_3 at which the structure function $S(l,x_3)$ is being measured.

will be superimposed mother eddies generated above x_3 and which will have therefore size $l_\rho>l_0\sim x_3$. Also these produce cascades superimposed on the original one, with daughter eddies of size $l_{\rho\sigma}<l_\rho$, indicating

$$l_{\rho\sigma}=e^{-\sigma}l_\rho=e^{\rho-\sigma}x_3. \tag{4}$$

It is clear that $\sigma\geq 0$, but it is also true that $\rho\geq 0$. This last condition means that, at height x_3, only mother eddies of size $l_\rho\geq x_3$ are present; smaller ones are generated at lower values of x_3.

In the presence of a Kolmogorov cascade, one will have that the typical ratio of the velocity inside daughter and mother eddies will be proportional to $(l_{\rho\sigma}/l_\rho)^{1/3}\exp(-r/l_{\rho\sigma})$, with $r\simeq(x_3 r_0^3)^{1/4}$ the viscous scale at height x_3. One can then write for the velocity difference $\mathbf{v_l(x)}\equiv\mathbf{v(x+l)}-\mathbf{v(x)}$:

$$\mathbf{v_l}(\mathbf{x})\sim\sum_i\left(\frac{l_{\rho_i\sigma_i}}{l_{\rho_i}}\right)^{1/3}\exp(-r/l_{\rho_i\sigma_i})\mathbf{u_l}(\mathbf{x},\mathbf{x}_i',\rho_i,\sigma_i), \tag{5}$$

where $\rho<R=\ln(\delta/x_3)$, and $\mathbf{u}(\mathbf{x},\mathbf{x}',\rho,\sigma)$ is the normalized velocity at position $\mathbf{x}$, due to a vortex of type $\rho\sigma$ centered at $\mathbf{x}'$ ($\mathbf{u_l}$ indicates finite difference with respect to the first argument).

An assumption on the statistics of $\mathbf{u}(\mathbf{x},\mathbf{x}',\rho,\sigma)$ becomes therefore necessary. Spatial correlations in this model are assumed to arise from the finite extension of individual eddies, while distinct eddies are taken to be uncorrelated. This leads to the expression

$$\langle u(\mathbf{x},\mathbf{x}'\rho,\sigma)u(\mathbf{y},\mathbf{y}',\rho',\sigma')\rangle$$

$$=v_*^2\delta(\rho-\rho')\delta(\sigma-\sigma')\delta(\mathbf{x}'-\mathbf{y}')C\left(\frac{|\mathbf{x}-\mathbf{y}|}{l_{\rho\sigma}}\right), \tag{6}$$

with $C(l_{\rho\sigma}^{-1}|\mathbf{x}-\mathbf{x}'|)$ the normalized $[C(0)=1]$ autocorrelation for a single vortex. In general, vortices will be distributed with a density $n(\mathbf{x},\rho,\sigma)l_{\rho\sigma}^{-(3-\zeta_{\rho\sigma})}$, with $\zeta_{\rho\sigma}$ allowing for the inclusion of intermittency corrections in the model. However, if $r_0\ll l_{\rho\sigma}\ll l_\rho\ll\delta$, scale invariance implies $n=n(l_\rho^{-1}x_3)$, with $n(1)\sim 1$, while $\zeta=0$ for locally space filling (nonintermittent) turbulence. Using Eq. (6), the two-point structure function can be computed explicitly:

$$S(l,x_3)=v_*^{-2}\langle v_l^2\rangle\sim\int_0^R d\rho\int_0^\infty d\sigma F(\exp[\ln(l/x_3)-\rho+\sigma])\exp\left[-\frac{2}{3}\sigma-\left(\frac{r_0}{x_3}\right)^{3/4}e^{3/4(\sigma-\rho)}\right], \tag{7}$$

where $F(\alpha)=1-C(\alpha)$. Assuming smoothness of the velocity profile in an individual vortex and finiteness of its total energy leads to the conditions on F:

$$F(\alpha)=O(\alpha^2)\quad(\alpha\ll 1),$$
$$\lim_{\alpha\to\infty}\alpha^2[F(\alpha)-1]=0. \tag{8}$$

For $\mathrm{Re}\to\infty$, one can then obtain asymptotic expressions for $S(l,x_3)$. For $l\ll x_3\ll\delta$, the integrals are dominated by the contribution at $\rho=0$ and $\sigma=\ln(x_3/l)$, i.e., the contribution from the eddies down the cascade generated at x_3, which have size l. In this way, Kolmogorov scaling arises:

$$S(l,x_3)=a\left(\frac{l}{x_3}\right)^{2/3}+O\left(\left(\frac{l}{\delta}\right)^{2/3},\left(\frac{l}{x_3}\right)^2\right),\quad l\ll x_3\ll\delta. \tag{9a}$$

For $l\gg x_3$ the dominant contribution is from mother eddies with size $x_3\leq l_\rho<\min(l,\delta)$. Due to the uniform distribution in ρ, this leads to a logarithmic form for the structure function:

$$S(l,x_3)=\frac{3}{2}\ln(l/bx_3)+O\left(\left(\frac{l}{\delta}\right)^{2/3}\right),\quad x_3\ll l\ll\delta, \tag{9b}$$

$$S(l,x_3)=\frac{3}{2}\ln(\delta/bx_3)+\epsilon(x_3/l),\quad x_3\ll\delta\ll l, \tag{9c}$$

with $\epsilon(x_3/l)$ at most $O((x_3/l)^2)$. The coefficients a and b and the various higher order terms in Eqs. (9a)–(9c), all depend on the detailed shape of the function $F(\alpha)$. The one-dimensional energy spectrum [25] is $E_k(x_3)=\int_{-\infty}^{+\infty}[S(\infty,x_3)-S(l,x_3)]e^{ikl}dl\sim|\partial S(k^{-1},x_3)/\partial k|$ in the inertial range. Hence, one has a standard $k^{-5/3}$ scaling for $l\ll x_3$ and a k^{-1} scaling when $x_3\ll l\ll\delta$. A definite expression for the energy spectrum can be obtained fixing the form of the autocorrelation. Taking $C(\alpha)=\exp(-\alpha^2)$, one obtains

$$E_k(x_3)=\pi^{1/2}v_*^2x_3\int_0^R d\rho\int_0^\infty d\sigma\exp\left[\rho-\frac{5}{3}\sigma-\frac{(kx_3)^2}{4}e^{2(\rho-\sigma)}-\left(\frac{r_0}{x_3}\right)^{3/4}e^{3/4(\sigma-\rho)}\right]. \tag{10}$$

A plot of this spectrum for different values of x_3/δ is shown in Fig. 2. A fit of wind tunnel data, taken from [26], is shown in Fig. 3; the data correspond to values of the ratios $x_3/\delta\simeq 0.01$ and $r_0/\delta\simeq 0.001$, i.e., to the experiment with $\mathrm{Re}_\theta=7076$ and $y^+=28$ illustrated in Fig. 1 of that reference.

As discussed in [26], v_l samples in the range $l\gg x_3$, velocities corresponding to vortices generated much above x_3, while for $l\ll x_3$, it samples the daughter eddies of size l generated in the cascade started at x_3. In [27], a k^{-1} scaling was obtained by means of a hierarchy of hairpin vortices of

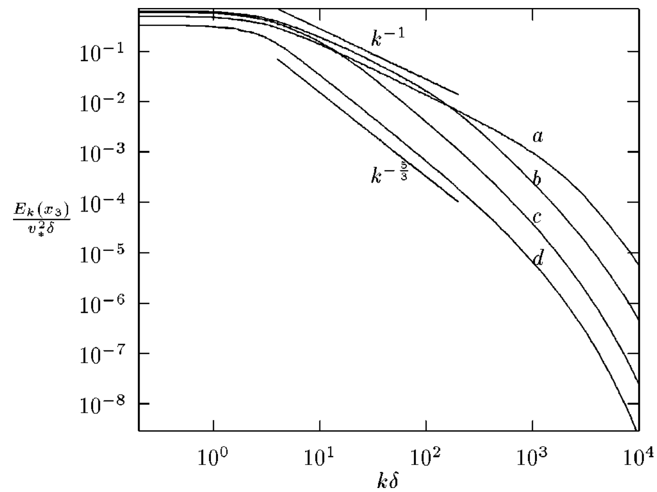

FIG. 2. One-dimensional energy spectra for four different values of x_3/δ. a, $x_3/\delta=0.001$; b, $x_3/\delta=0.01$; c, $x_3/\delta=0.1$; d, $x_3/\delta=0.5$. In all cases, $\delta/r_0\simeq 10^5$.

prescribed shape, but the transition to the $k^{-5/3}$ range was left out of the description. This phenomenon is explained here as a natural consequence of implementing a Kolmogorov cascade in an inhomogeneous turbulence setting; in all cases, hairpin vortices, and coherent structures in general, do not seem to be essential in obtaining such scaling behaviors.

The range $l\gg x_3$ is where turbulence ceases to be homogeneous and isotropic. Notice that the divergence of $\langle v_1^2\rangle$ as $x_3/\delta\to 0$ in Eq. (9c) forces an implicit assumption of anisotropy in the model, in order to avoid inconsistency with Eq. (2). Thus, although $\langle v_1^2\rangle$ diverges as $x_3/\delta\to 0$, $\langle v_1v_3\rangle$ remains equal to v_*^2; this result can be obtained assuming that the velocity of vortices generated much above x_3 is almost parallel to the wall at x_3. Hence, while $\langle v_1^2\rangle$ receives contributions from all vortices, up to size δ, $\langle v_1v_3\rangle$ receives contribution only from vortices up to size x_3.

Although essentially kinematic, this model is dynamically consistent. In spite of the fact that all cascades overlap in space, the dominant interactions appear to be those among

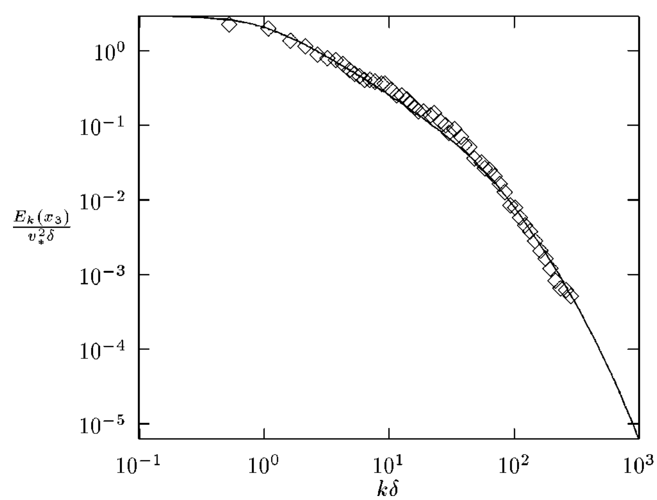

FIG. 3. Fit of wind tunnel data for $x_3/\delta\simeq 0.01$ and $r_0/\delta\simeq 0.001$. The curve is shifted to the left to overlap with the data, corresponding to a shift by $\simeq\ln(3)$, of the limits of integration in ρ, in Eqs. (7) and (10). Physically, this would correspond to a transition to anisotropy at $l\simeq 3x_3$.

vortices of similar size, as in standard Kolmogorov theory, and belonging to the same cascade. A rough argument could be the following. The relevant strain over an eddy of size l, from a cascade starting at height x_3, is produced by eddies of size $l' \geqslant l$. This strain will be of the order of: $v_* l'^{-1}(l'/x_3')^{1/3}$ with x_3' the height of generation of the second cascade. If $x_3' < x_3$, however, the volume in which the interaction takes place will be reduced by x_3'/x_3 and the effective strain will be of the order of $\min(1,x_3'/x_3)v_* l'^{-1}(l'/x_3')^{1/3}$. The effective strain will be maximum then for $x_3 = x_3'$ and $l = l'$ corresponding to interaction with vortices of the same size and belonging to the same cascade.

III. THE CASE OF "INFINITE SPACE" CONVECTIVE TURBULENCE

Stratification in a fluid produces buoyant forces that couple velocity and temperature in the Navier-Stokes equation. In the limit of weak stratification in a large volume, one can adopt the Boussinesq approximation [5], in which the buoyant force in the Navier-Stokes equation, and the production term in the temperature equation, are linearized respectively in the temperature and the vertical velocity fluctuation. For a unitary density medium:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = \nabla P + \nu\nabla^2\mathbf{v} - g\Theta^{-1}e_3\theta, \qquad (11a)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\theta = \sigma\nabla^2\theta + \Theta' v_3. \qquad (11b)$$

Here, σ is the molecular diffusivity, g is the gravitational acceleration, θ and Θ are the fluctuating and mean potential temperature, while $\Theta' \equiv d\Theta/dx_3$. The equations are linearly stable for $\Theta' > 0$; in this case, external forcing is necessary to achieve a stationary turbulent state.

In homogeneous isotropic turbulence, one derives Kolmogorov scaling [2], using dimensional analysis with the quantities that are available: the scale l, the velocity difference v_l and the mean (kinetic) energy dissipation ϵ; this leads to the well-known result $\langle v_l^2\rangle \sim (\epsilon l)^{2/3}$. In the case of a stably stratified medium, in the presence of mechanical forcing, Bolgiano [22,23] hypothesized an alternative situation, in which, potential energy transfer due to buoyancy forces, rather than kinetic energy transfer, governs turbulence dynamics. Dimensional reasoning with the quantities available in Eqs. (11a) and (11b), leads then to the scalings [8]

$$\langle v_l^2\rangle = C_{vv}\left(\frac{g}{\Theta}\right)^{4/5}\epsilon_\theta^{2/5} l^{6/5}, \quad \langle v_l\theta_l\rangle = C_{v\theta}\left(\frac{g}{\Theta}\right)^{1/5}\epsilon_\theta^{3/5} l^{4/5},$$

$$\langle \theta_l^2\rangle = C_{\theta\theta}\left(\frac{g}{\Theta}\right)^{-2/5}\epsilon_\theta^{4/5} l^{2/5}, \qquad (12)$$

where ϵ_θ is the dissipation of temperature fluctuations:

$$\epsilon_\theta = \sigma|\nabla\theta|^2. \qquad (13)$$

It is interesting to carry out this dimensional reasoning, directly inside Eqs. (11a) and (11b). After introducing the eddy turnover frequency $\omega_l \sim l\langle v_l^2\rangle^{1/2}$ and considering scales l much larger than the dissipation lengths for $\mathbf{v}$ and θ, we have from Eq. (11a):

$$\omega_l\langle v_l^2\rangle \sim \frac{g}{\Theta}\langle v_l\theta_l\rangle \quad \text{and} \quad \omega_l\langle v_l\theta_l\rangle \sim \frac{g}{\Theta}\langle \theta_l^2\rangle, \qquad (14)$$

while, from Eq. (11b):

$$\omega_l\langle v_l\theta_l\rangle \overset{?}{\sim} \Theta'\langle v_l^2\rangle \quad \text{and} \quad \omega_l\langle \theta_l^2\rangle \sim \epsilon_\theta \overset{?}{\sim} \Theta'\langle v_l\theta_l\rangle. \qquad (15)$$

The question marks in Eq. (15) indicate places in which the relation between terms is ambiguous. It appears that the ambiguity lies in the source term $\Theta' v_3$ in Eq. (11b). One sees immediately that the scaling described in Eq. (12) is obtained when the term in Θ' is negligible in Eqs. (11b) and (15). Introducing the Obukhov length

$$L_* = \epsilon_\theta^{1/2}\Theta'^{-5/4}\left(\frac{\Theta}{g}\right)^{1/4}, \qquad (16)$$

one realizes from Eqs. (12) and (15) that the condition of the negligible Θ' term is equivalent to $l \ll L_*$. Thus, one has a source of temperature fluctuations at $l \leqslant L_*$, whose energy is transferred to smaller scales by action of the convective term $\mathbf{v}\cdot\nabla\theta$. These fluctuations provide the forcing for the velocity, through the buoyant term $ge_3\theta/\Theta$, in Eq. (11a).

In order for a cascade of this form to be present, the transfer of kinetic energy must be negligible, or equivalently, the frequency ω_l (which is the strain felt by eddies at scale l) must dominate the one that would be produced in a Kolmogorov cascade, indicating with ϵ the rate of kinetic energy production at the largest scales:

$$\omega_l \sim v_{L_*}L_*^{-3/5}l^{-2/5} \gg \epsilon^{1/3}l^{-2/3} \sim v_{L_*}L_*^{-1/3}l^{-2/3}, \qquad (17)$$

which implies $l \gg L_*$. Thus, the two conditions of negligible Θ' in Eq. (15) and dominant buoyant force in Eq. (14) restrict the possibility of Bolgiano scaling at most to a finite range around L_*.

An alternative situation, in which kinetic energy flows towards large scales, has been considered in [28]. In this case, the condition provided by Eq. (17) is not necessary anymore; however, the condition of negligible Θ' still forces Bolgiano scaling, to the range $l < L_*$. Now, this is the range in which the equation for the velocity decouples from that for the temperature, and it is difficult to see a mechanism whereby buoyancy could modify the nonlinear interaction in such a way as to invert the direction of the energy transfer. If one restricts to this range, and maintains a situation of forward energy transfer, the decoupling forces the temperature to be advected like a passive scalar, which results in the well-known Kolmogorov-Corrsin scaling [29]:

$$\langle v_l^2\rangle = C_{vv}(\epsilon l)^{2/3}, \quad \langle v_l\theta_l\rangle = C_{v\theta}\epsilon_\theta^{1/2}\epsilon^{1/6}l^{2/3},$$

$$\langle \theta_l^2\rangle = C_{\theta\theta}\epsilon_\theta\epsilon^{-1/3}l^{2/3}. \qquad (18)$$

The only way to obtain a power law fluctuation spectra for $l > L_*$ remains then, that

$$\langle v_l \theta_l \rangle \sim g^{-1} \Theta \epsilon \sim \Theta'^{-1} \epsilon_\theta . \tag{19}$$

This would lead again to a $l^{2/3}$ spectrum for $\langle v_l^2 \rangle$ and $\langle \theta_l^2 \rangle$; in this case, however, there would be a privileged scale, the Obukhov length, which would fix the amplitude of the cross correlation $\langle v_l \theta_l \rangle$:

$$\frac{\langle v_l \theta_l \rangle}{(\langle v_l^2 \rangle \langle \theta_l^2 \rangle)^{1/2}} \sim \left(\frac{L_*}{l} \right)^{2/3} . \tag{20}$$

This leads to the situation of correlations between v_l and θ_l being the strongest at $l \sim L_*$ and decaying at larger scales.

From these observations, it is clear that scaling behaviors should not be expected for $l > L_*$. An interesting question is then how a toy system simulating a large range of scales like a GOY model would behave in the presence of buoyancy. GOY models (see [30] and references therein) present a cascade of energy along a linear chain of coupled ordinary differential equations for the complex variables u_n, which are the analog of the velocity of eddies at scales $l_n = 2^{-n}$ in real turbulence. These models have attracted great attention, due to the coincidence of the intermittent properties of the moments $\langle |u_n|^n \rangle$ and those of the structure functions $\langle v_l^n \rangle$ in real turbulence. Jensen *et al.* [21] have derived a generalization to the case of a passive scalar advected by a turbulent velocity field. It is easy to include in their model the effect of buoyancy; the resulting set of equations reads ($k_n = l_n^{-1}$)

$$\begin{aligned}
(\partial_t - \nu k_n^2) u_n &= i k_n (a u_{n+1} u_{n+2} + b u_{n-1} u_{n+1} + c u_{n-1} u_{n-2})^* \\
&\quad - \alpha T_n + f_n , \\
(\partial_t - \sigma k_n^2) T_n &= i k_n [e_n (u_{n-1} T_{n+1} + u_{n+1} T_{n-1}) \\
&\quad + g (u_{n-2} T_{n-1} + u_{n-1} T_{n-2}) \\
&\quad + h (u_{n+1} T_{n+2} + u_{n+2} T_{n+1})]^* + \beta u_n
\end{aligned} \tag{21}$$

for $n = 1,2,\ldots,N$, where the parameters a,b,c,e,g,h are given by

$$a = 1, \quad b = c = g = -e = -h = -\frac{1}{2}; \tag{22}$$

and variables u_m and T_m in the nonlinearities are set identically equal to zero for $m < 1$ and $m > N$. The choice here is opposite to that of [28], in the sense that it has been preferred not to tamper with the nonlinearities, and to leave them in the same form as without buoyancy. However, also in this case, some interesting results are obtained.

Equations (21) and (22) have been integrated numerically for both stable and unstable conditions. In the stable case, a constant forcing $f_n = (1+i) \times 10^{-3} \delta_{n4}$ has been used to sustain fluctuations.

In the unstable case no external forcing was present, and the fluctuations organized in such a way to push the Obukhov scale $N_* = \log_2(\epsilon_T^{-1/2} \beta^{5/4} \alpha^{-1/4})$ towards the lowest available shell; a Kolmogorov cascade ensued in all cases (see Fig. 4).

In the stable case, the presence of two additional parameters with which to play, the forcing amplitude and wave

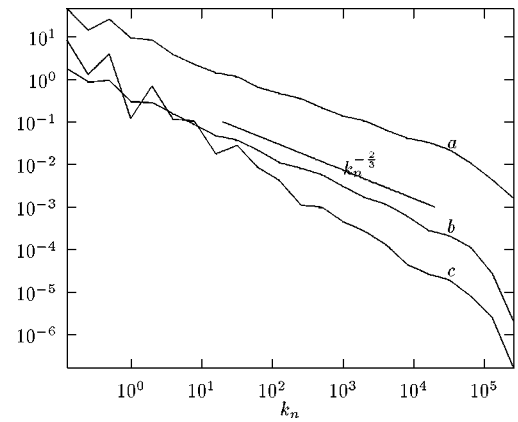

FIG. 4. GOY model simulation of convective turbulence under unstable conditions; values of the parameters are $\alpha = 0.01$, $\beta = -0.2$, $\nu = \sigma = 10^{-7}$; no external forcing. a: $\langle |T_n|^2 \rangle$; b: $\langle |u_n|^2 \rangle$; c: $\langle |u_n T_n| \rangle$. The buoyancy terms become of the same order of the others only for $k_n \leq 1$.

number, allowed better control of N_*. When N_* was sufficiently large, neither Bolgiano scaling nor the situation depicted in Eq. (18) took place, rather, a combination of the two, with steep $\sim k^{-1}$, overlapping T and u spectra, and an almost constant cross correlation $\langle u_n T_n \rangle$ (see Fig. 5).

An attempt has been carried also to generate a backward kinetic energy cascade, using small scale forcing (the forced shell was $n = 20$). A physical interpretation of such a condition could be the presence of plumes generated elsewhere in the fluid, in some steep thermal layer. Using Eqs. (21) and (22), the outcome was, however, a $k^{-5/3}$ spectrum in all situations.

One can compare the behavior of N_* in the two stability situations, looking at the ratios of the fluctuation amplitudes of the buoyancy terms αT_n and βu_n, to those of $\partial_t u_n$ and $\partial_t T_n$. One sees in Fig. 6 how in the stable case, buoyancy remains dominant over the whole inertial range, while the nonlinearity dominates the dynamics in the unstable case.

In nature, of course, things go differently. First of all, the largest available scale l_0 corresponds to the size of the system. In the unstable case, steep thermal boundary layers develop rapidly and an approximation of constant temperature

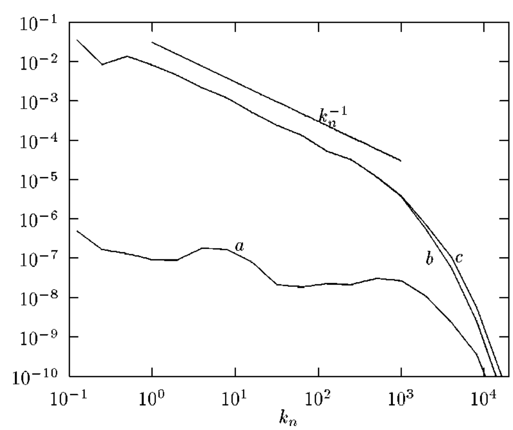

FIG. 5. GOY model simulation of convective turbulence under stable conditions; values of the parameters are $\alpha = \beta = 10$; $\nu = \sigma = 10^{-7}$, $f_n = 10^{-3}(1+i)\delta_{n4}$. a: $\langle |u_n T_n| \rangle$; b: $\langle |u_n|^2 \rangle$; c: $\langle |T_n|^2 \rangle$. The buoyancy terms are of the same order of the others, over the whole range of k_n, down to the dissipation scale.

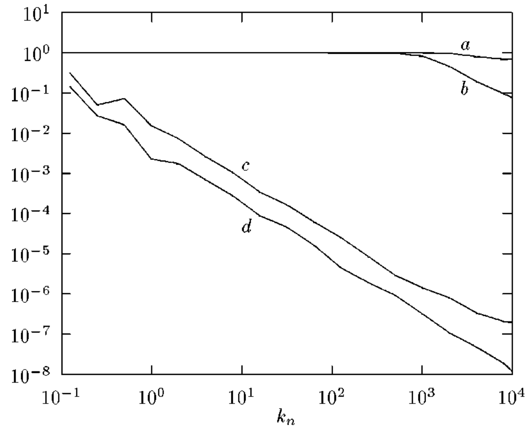

FIG. 6. Plots of the ratios: $r_1=\langle|\alpha T_n|^2\rangle/\langle|\partial_t u_n|^2\rangle$ and $r_2=\langle|\beta u_n|^2\rangle\langle|\partial_t T_n|^2\rangle$ vs k_n. a: r_1; stable. b: r_2; stable. c: r_1; unstable. d: r_2; unstable.

gradient ceases to be applicable. Thus, the left portion of the spectra in Fig. 4 is not particularly meaningful, and the prediction that $L_*\to l_0$ under unstable conditions should not be trusted. In fact, in convective atmospheric turbulence, one has $L_*<l_0$, and a large scale, buoyancy dominated range is indeed present [31].

Same amount of difficulty occurs in the treatment of stable environments. In this case the problem is the idea of a purely large scale mechanical forcing. It is well known that, in these conditions, the large scale modes in turbulent shear layers are stabilized by buoyancy; thus, the peak in the forcing is moved to $l\sim L_*$, and a range like that of Fig. 4 is not generated [31]. However, a purely large scale forcing can be generated, if low frequency waves are present in the flows; in this case, a turbulent spectrum extending to the buoyancy dominated region $l<L_*$ becomes possible again (see, e.g., [32]).

IV. POSSIBILITY OF A PURELY HELICAL CASCADE

The last situation that is taken into consideration is that of non-reflection-invariant turbulence. It is well known that, beyond energy, the nonlinearity of the Navier-Stokes equation has a second global invariant, the total helicity:

$$H=\frac{1}{2}\int d^3x\mathbf{v}(\mathbf{x})\cdot[\nabla\times\mathbf{v}(\mathbf{x})]. \tag{23}$$

In many ways, helicity is the counterpart in three dimensions of two-dimensional vorticity, and a natural question to ask is whether three-dimensional turbulence may exhibit multiple cascades, as it happens in two dimensions. The ability of helicity to hinder energy transfer [33,10], in particular, suggests the possibility of a helicity cascade with no energy transfer, given appropriate conditions on the forcing.

Helicity, however, has the peculiarity of being a nonpositive defined pseudoscalar. Lack of positive definiteness implies, in particular, that any triad of interacting modes can exchange helicity in an arbitrary way, thus providing a source of helicity transfer fluctuations.

This effect turns out to be important in GOY models; even in the case of maximum injection of helicity, for a given energy injection rate, it can be shown that the amount of the GOY equivalent of helicity: $H=\Sigma_n(-k_n)^n|u_n|^2$ [20], which is produced by fluctuations, is much greater than the one coming from forcing [34]. Since the variable u_n in GOY models mimics in a surprising way the velocity inside individual scale l_n eddies, this may be a serious indication of the impossibility of a helicity cascade.

Anyway, such a cascade seems impossible, also in a purely ''mean field'' description, with a helicity transfer to small scales, assumed constant over the whole space.

The standard sequence of arguments, leading to Kolmogorov scaling, can be carried out, assuming a constant helicity flux ϵ_H to small scales; indicating with $H_l\sim l^{-1}v_l^2$ the content of helicity at scale l one can then write

$$\epsilon_H\sim\omega_lH_l\sim l^{-2}v_l^3=\text{const.}\Rightarrow v_l\sim\epsilon_H^{1/3}l^{2/3} \tag{24}$$

implying expressions for the energy and helicity spectra and for the eddy turnover frequency:

$$E_k=c_1\epsilon_H^{2/3}k^{-7/3},\quad H_k=c_2\epsilon_H^{2/3}k^{-5/3},\quad \omega_k=c_3\epsilon_H^{1/3}k^{1/3}. \tag{25}$$

It is possible to obtain energy and helicity balance equations using statistical closure, starting from the expression for the velocity correlation $U_{\mathbf{k}}^{ij}=\langle|v_{\mathbf{k}}^iv_{-\mathbf{k}}^j|\rangle$:

$$2\pi U_{\mathbf{k}}^{ij}=k^{-2}P^{ij}(\mathbf{k})E_k+k^{-4}\epsilon^{ijl}k^lH_k,\quad P^{ij}(\mathbf{k})=\delta^{ij}-\frac{k^ik^j}{k^2}. \tag{26}$$

Lesieur has derived such balance equations within the EDQNM closure [10]. In this kind of closure [24,10], the third order correlations $\langle vvv\rangle$, which enter the equation for U_k^{ij}, are approximated by: $\langle vvv\rangle\simeq\langle v^{(1)}v^{(0)}v^{(0)}\rangle+\langle v^{(0)}v^{(1)}v^{(0)}\rangle+\langle v^{(0)}v^{(0)}v^{(1)}\rangle$, with $v^{(1)}$ the first order perturbative solution to a modified Navier-Stokes equation, with the viscous term νk^2 replaced by the eddy turnover frequency ω_k. The $v^{(0)}$ are taken uncorrelated, so that the resulting 4-point correlations split into products of 2-point correlations. Furthermore, the approximation $\langle|v_{\mathbf{k}}^i(t)v_{-\mathbf{k}}^j(0)|\rangle\simeq U_{\mathbf{k}}^{ij}\exp(-\omega_k|t|)$ is adopted.

The EDQNM equations for the energy and helicity balance read therefore [10]

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)E_k=\int_{\Delta_k}dpdq\,\theta_{kpq}\left[\frac{k}{pq}b_{kpq}E_q(k^2E_p-p^2E_k)-\frac{1}{p^2q}c_{kpq}H_q(k^2H_p-p^2H_k)\right] \tag{27a}$$

and

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)H_k=\int_{\Delta_k}dpdq\,\theta_{kpq}\left[\frac{k}{pq}b_{kpq}E_q(k^2H_p-p^2H_k)-\frac{k^2}{q}c_{kpq}H_q(k^2E_p-p^2E_k)\right], \tag{27b}$$

where

$$\theta_{kpq}=(\omega_k+\omega_p+\omega_q)^{-1},\quad b_{kpq}=\frac{p}{k}(xy+z^3),$$

$$c_{kpq}=z(1-y^2) \tag{28}$$

with Δ_k the domain in which k, p, and q can be the lengths of the sides of a triangle, and x, y, and z the cosines of the angles opposite to these sides.

The condition $\epsilon_H=\text{const}$ in Eqs. (24) and (25) and conservation of helicity triad by triad guarantee that the spectra and frequencies of Eq. (25) provide a stationary solution for Eq. (27b) for any value of the coefficients c_i. The energy balance, which is given by Eq. (27a) fixes instead, at stationarity, the ratio of the two coefficients c_1 and c_2. Numerical integration of that equation leads then to the result

$$\frac{c_2}{c_1}\simeq 3.316. \tag{29}$$

However, from the definition of helicity, one has

$$k^{-1}|H_k|=2\pi k\langle|[i\mathbf{k}\times\mathbf{v_k}]\cdot\mathbf{v}_{-\mathbf{k}}|\rangle\leqslant E_k=2\pi k^2\langle|v_\mathbf{k}|^2\rangle, \tag{30}$$

which implies $|c_2|\leqslant c_1$. Thus, Eq. (29) cannot be satisfied, and a helicity cascade of the type described by Eqs. (24) and (25) does not seem to be possible.

V. CONCLUSIONS

The aim of this paper was to obtain some information on the effect of large scale flow inhomogeneities on the form of the energy spectra in turbulent fluids. Some idealized situations have been studied by means of simplified models and closure analysis. The point in common in the three inhomogeneous turbulence situations considered is that simple dimensional reasoning either gives wrong answers or does not lead to any answer at all. Of course this was something to be expected and, in a certain sense, there is nothing deep in this result. However, the practical consequences are important.

The analysis carried on here clearly shows that a k^{-1} range is a universal feature of mechanical turbulent layers, which is independent of the presence of coherent structures. If one is interested in diffusion in wall turbulence situations, a k^{-1} range at scales $kx_3<1$ clearly makes a difference, with respect to a $k^{-5/3}$ spectrum, extending down to the inverse of the boundary layer thickness. Particles at distance $l>x_3$ will separate horizontally, in almost a ballistic way: $l(t)\sim t$ (with logarithmic corrections), while Richardson law $l_3(t)\sim t^{3/2}$ will dominate in the vertical direction. The modifications that would be produced in dispersion models, for situations in which turbulence is predominantly mechanical, are clearly worth investigating.

The interest in the existence of Bolgiano scaling is more academic, although some application to turbulent boundary layers in stable environments, in the presence of forcing by low frequency waves, is possible [32]. This scaling attracted some interest a few years ago to explain observations carried on in liquid helium convection experiments [35]. This approach has been criticized later by several authors [18,36]. The result of the analysis carried on here suggests analogous difficulties for the existence of Bolgiano scaling in an idealized situation of convective turbulence in an infinite volume. The alternative, however, which is characterized by velocity and temperature $k^{-5/3}$ spectra, has difficulties itself due to the presence of a privileged scale, the Obukhov length, dominating the dynamics, which weakens the very concept of an inertial range. GOY model simulations suggest indeed that neither scaling should be observed, rather, under stable conditions, a k^{-2} spectrum for both temperature and velocity, with correlation between the two, vanishing at large scales, should develop.

The possibility of helical turbulence has sparked recently some attention in people interested in turbulence control [37]. A $k^{-7/3}$ energy spectrum, associated with a helicity cascade, would imply a decrease in the energy dissipation of the order of Re^{-1} with respect to the standard $k^{-5/3}$ situation. (For equal total turbulent energy in the two cases, $\epsilon_H\sim\epsilon L^{-1}\Rightarrow\epsilon'\sim l_d\epsilon_H$ with ϵ_H and ϵ', respectively, the helicity and energy dissipation for a $k^{-7/3}$ situation, ϵ the energy dissipation for the corresponding $k^{-5/3}$, and L and l_d, respectively, the integral and viscous scales.) The impossibility of a helicity cascade, suggested by the EDQNM calculation carried on here, means simply that a $k^{-7/3}$ could not be obtained by modifying the large scale forcing, and that action at all scales (or alternatively at all frequency) would be necessary. Hence, turbulence control by forcing the cascade to become helicity dominated could be possible only using a feedback system acting at inertial range frequencies.

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